Volume Distance to Hypersurfaces: Asymptotic Behavior of its Hessian

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Abstract. The volume distance from a point p to a convex hypersurface $M \subset \mathbb{R}^{N+1}$ is defined as the minimum $(N+1)$-volume of a region bounded by M and a hyperplane H through the point. This function is differentiable in a neighborhood of M and if we restrict its hessian to the minimizing hyperplane $H(p)$ we obtain, after normalization, a symmetric bi-linear form Q.

In this paper, we prove that Q converges to the affine Blaschke metric when we approximate the hypersurface along a curve whose points are centroids of parallel sections. We also show that the rate of this convergence is given by a bilinear form associated with the shape operator of M. These convergence results provide a geometric interpretation of the Blaschke metric and the shape operator in terms of the volume distance.

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1. Introduction

Consider a strictly convex hypersurface $M \subset \mathbb{R}^{N+1}$, a point p in the convex side of M and $n \in S^N$. Denote by $U(n,p)$ the region bounded by M and a hyperplane $H(n,p)$ orthogonal to n through p, with n pointing outwards the region, and by $V(n,p)$ its volume. The *volume distance* $v(p)$ of p to M is defined as the minimum of $V(n,p)$, $n \in S^N$.

The volume distance is an important object in computer vision which has been extensively studied in the planar case $n = 1$ ([1]) and was also considered in the case $n = 2$ ([4]). For $n = 1$, the hessian of the volume distance was studied in ([2],[3]), where it is shown that its determinant equals -1. This property is not extended to higher dimensions. Nevertheless, we

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prove in this paper some asymptotic properties of the hessian of the volume distance in arbitrary dimensions.

A pair (n,p) is called *minimizing* when n is the minimum of $V(n,p)$ with p fixed. A minimizing pair necessarily satisfies

$$\frac{\partial V}{\partial n}(n,p)=0. \tag{1.1}$$

It is proved in [5] that if (n,p) satisfies (1.1), then p is the centroid of $R(n,p)$.

In order to obtain $n=n(p)$ implicitly defined by (1.1), the second derivative of V with respect to n must be non-degenerate. A formula for this second derivative can also be found in [5]. From this formula, one concludes that the second derivative is positive definite in a half-neighborhood of M, i.e., the part of a neighborhood of M contained in its convex side. Based on this, we verify that there exists a half-neighborhood D of M such that, for any $p\in D$, there exists a unique $n(p)$ that minimizes the map $n\to V(n,p)$. Moreover, the map $p\to n(p)$ is smooth and consequently $v(p)=V(n(p),p)$ is also smooth.

For $p\in D$, let

$$Q(p)=\frac{1}{b(p)}\frac{\partial^2 V}{\partial n^2}(n(p),p), \tag{1.2}$$

where $b(p)$ denotes the N-dimensional volume of the region $R(p)\subset H(p)$ bounded by M. By making some calculations, we show that, for $p\in D$,

$$-\frac{1}{b(p)}\left.D^2v(p)\right|_{H(p)}=Q^{-1}(p) \tag{1.3}$$

where $\left.D^2v(p)\right|_{H(p)}$ means the restriction of $D^2v(p)$ to $H(p)$.

This paper is concerned with the asymptotic behavior of the quadratic form Q. In order to motivate a bit more this study, we remark that this quadratic form is an important tool in the study of *floating bodies*. When M is the boundary of a convex body K, one can define its floating body K_δ, for $\delta>0$, by the property that each support hyperplane of K_δ cuts K in a region of volume δ. For smooth strictly convex bodies and δ sufficiently small, the convex bodies exist and its boundary is a smooth surface (see [5]). In [6], the quadratic form Q was a key ingredient in proving that K_δ is well defined for every $0<\delta\leq\frac{1}{2}vol(K)$ if and only if K is symmetric with respect to a point. Also in [9], Q appears as a tool in proving that a convex body with a sequence of homothetic floating bodies must be an ellipsoid.

For $q\in M$, denote by $T_qM=H(n(q),q)$ the tangent plane to M at q and, for $t>0$, define $\gamma_q(t)$ as the centroid of the region $R(n(q),q+t\xi(q))$, where $\xi(q)$ is the affine normal to M at q. We shall consider two symmetric bilinear forms defined on T_qM: the Blaschke metric h which is positive definite and h_S defined as $h_S(X,Y)=h(X,SY)$, where S is the shape operator. By identifying $H(\gamma_q(t))$ with T_qM, the normalized hessian $Q(\gamma_q(t))$ can also be seen as a symmetric bilinear form in T_qM. The main result of the paper says that

$$Q(\gamma_q(t))=h(q)+th_S(q)+O(t^2),$$

where $O(t^k)$ indicates a quantity such that $\lim_{t\to 0}\frac{O(t^k)}{t^{k-\epsilon}}=0$, for any $\epsilon>0$. This result can be regarded as a geometric interpretation of the Blaschke metric and the shape operator in terms of the volume distance.

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2. Hessian of the volume distance

2.1. Notation

Consider a strictly convex hypersurface $M\subset\mathbb{R}^{N+1}$, possibly with a non-empty boundary ∂M. Denote by $H(n,p)\subset\mathbb{R}^{N+1}$ the hyperplane passing through $p\in\mathbb{R}^{N+1}$ with normal $n\in S^N$. For $p\in\mathbb{R}^{N+1}$, denote by $E(p)\subset S^N$ the set of unitary vectors n whose corresponding hyperplane $H(n,p)$ intersects $M-\partial M$ transversally at a closed hypersurface $\Gamma(n,p)\subset H(n,p)$ bounding a region $R(n,p)\subset H(n,p)$ containing p in its interior and such that the region $U(n,p)$ bounded by $R(n,p)$ and M, with n pointing outwards, has finite volume $V(n,p)$ (see figure 1). Denote by $D_1\subset\mathbb{R}^{N+1}$ the set of $p\in\mathbb{R}^{N+1}$ such that $E(p)\neq\emptyset$ and the infimum $\inf\{V(n,p)|\ n\in E(p)\}$ is attained at $E(p)$. When $n\in E(p)$ attains this minimum, we call the pair (n,p) minimizing and $v(p)=V(n,p)$ the volume distance to M. We remark that if M is a closed hypersurface enclosing a convex region, then the domain D_1 of the volume distance is all the enclosed region.

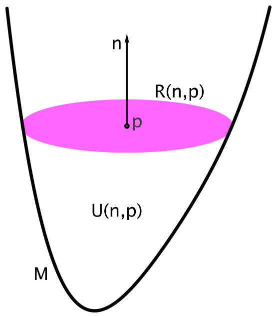

FIGURE 1. The section $R(n,p)$ and the enclosed region $U(n,p)$.

For $q\in M$, denote by $\xi(q)$ the affine normal vector pointing to the convex side of M. Along this paper, we shall call a half-neighborhood of M any set of the form

$$\{q+t\xi(q)|\ q\in M, 0\le t<T(q)\},$$

where $T(q)>0$ is some smooth function of q.

Close to a pair (n_0,p_0), consider cartesian coordinates $(x,z)\in\mathbb{R}^N\times I$, $I=(-\epsilon,\epsilon)$ such that $p_0=(0,0)$ and $n_0=(0,1)$. To describe the hypersurface

M in a neighborhood of $H(n_0,p_0)$, consider cylindrical coordinates (r,η,z), where $x=r\eta$, $\eta\in S^{N-1}$, $r>0$. Then M is described by $r=r(\eta,z)$, for some smooth function r (see figure 2). We write

$$r(\eta,z)=r(\eta,0)+r_z(\eta,0)z+O(z^2), \tag{2.1}$$

for z close to 0.

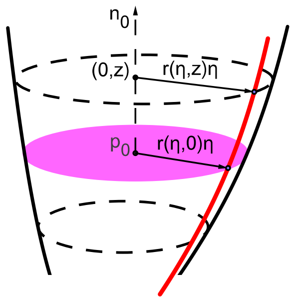

FIGURE 2. The curve $r=r(\eta,z)$ with fixed $\eta\in S^N$.

2.2. Smoothness of the volume distance v in a half-neighborhood of M

The derivative $\frac{\partial V}{\partial n}(n,p_0)$ can be regarded as a linear functional on T_nS^N, which can be identified with $H(n,p_0)$. The proof of next proposition can be found in [5], p. 166.

Proposition 2.1. *Denote by $\overline{p}(n,p)$ the center of gravity of $R(n,p)$ and by $b(n,p)$ the N-dimensional volume of the region $R(n,p)$. Then*

$$\frac{\partial V}{\partial n}(n,p)=-b(n,p)\left(\overline{p}(n,p)-p\right). \tag{2.2}$$

Thus, a pair (n,p) is critical if and only if $\overline{p}(n,p)=p$.

The second derivative $\frac{\partial^2 V}{\partial n^2}(n,p)$ can be seen as a linear operator of T_nS^N. Next proposition, whose proof can be found in [5], p. 168, describe this linear operator in the above defined cylindrical coordinates.

Proposition 2.2. *Denote $\mathbf{M}_N$ the symmetric positive definite $N\times N$ matrix $\eta\cdot\eta^t$, where η is a column vector and η^t its transpose. We have that*

$$\frac{\partial^2 V}{\partial n^2}(n_0,p_0)=\int_{S^{N-1}} r^{N+1}(\eta,0)r_z(\eta,0)\mathbf{M}_N d\eta. \tag{2.3}$$

If $r_z(\eta)>0$, for any $\eta\in S^{n-1}$, then formula (2.3) implies $\frac{\partial^2 V}{\partial n^2}(n_0,p_0)$ is positive definite. Based on this, we can prove the following proposition:

Proposition 2.3. *There exists a half-neigborhood $D \subset D_1$ of M such that for any $p \in D$ there exists a smooth function $n = n(p)$ such that the pair $(n(p), p)$ is minimizing and $\frac{\partial^2 V}{\partial n^2}(n(p), p)$ is positive definite.*

Proof. Given $q \in M$ consider a neighborhood W of q in M with the following property: for any pair (n, p) such that $\Gamma(n, p) \subset W$, $r_z(n, p)$ is strictly positive. For p fixed, denote by $E_1(p) = \{n \in S^{N-1} | \, \Gamma(n, p) \subset W\}$.

There is a half-neighborhood $U(q)$ of q such that for any $p \in U(q)$, there exists a minimizing $n(p) \in E_1(p)$ and any minimizing pair $n(p)$ must be in $E_1(p)$. Since $r_z(n, p)$ is strictly positive, the map $n \in E_1(p) \to V(n, p)$ is convex, so the minimizer $n(p)$ is unique. Considering $D = \cup_{q \in M - \partial M} U(q)$, we complete the proof of the proposition. □

2.3. Derivatives of the volume distance

Consider D the half-neighborhood of M given by proposition 2.3 and let $p \in D$. Recall that

$$v(p) = V(n(p), p). \tag{2.4}$$

Lemma 2.4. *We have that*

$$\frac{\partial V}{\partial p}(n, p) = b(n, p) n. \tag{2.5}$$

As a consequence,

$$Dv(p) = b(n(p), p) n(p). \tag{2.6}$$

Proof. Since $p \to V(n, p)$ is constant along the hyperplane $H(n, p)$, we conclude that $\frac{\partial V}{\partial p}(n, p)$ is parallel to n. Also, for t small,

$$V(n, p + tn) - V(n, p) = tb(n, p) + O(t^2),$$

and thus the first formula is proved. Now differentiating (2.4) we obtain (2.6). □

Proposition 2.5. *The normalized hessian of v is exactly Q^{-1}, i.e.,*

$$-\frac{1}{b(p)} \, D^2 v(p)\big|_{H(p)} = Q^{-1}.$$

Proof. Differentiating (2.6) with respect to p and using that n is orthogonal to $H(p)$, we obtain

$$D^2 v(p)\big|_{H(p)} = b(p) \left. \frac{dn}{dp} \right|_{H(p)}.$$

On the other hand, if we differentiate (1.1) with respect to p we obtain

$$\frac{\partial^2 V}{\partial n^2}(n, p) \frac{dn}{dp} + \frac{\partial^2 V}{\partial n \partial p} = 0.$$

Now, from (2.5),

$$\frac{\partial^2 V}{\partial n \partial p} = b(p) I + \frac{\partial b}{\partial n} n.$$

We conclude that

$$\left.\frac{dn}{dp}\right|_{H(p)} = -b(p)\left[\frac{\partial^2 V}{\partial n^2}(n,p)\right]^{-1},$$

thus proving the proposition. □

3. Convergence to the Blaschke metric

For $q \in M$, consider the centroid $\gamma_q(t),\ t>0$ of the region $R(n(q), q+t\xi(q))$, where $n(q)$ is orthogonal to T_qM and $\xi(q)$ is the affine normal vector at q. Then $Q(\gamma_q(t))$ is a symmetric bilinear form defined in $H(\gamma_t(q))$, which can be identified with T_qM. The aim of this section is to prove the following theorem:

Theorem 3.1. *For $q \in M$,*

$$Q(\gamma_q(t)) = h(q) + O(t), \tag{3.1}$$

and so $Q(\gamma_q(t))$ is converging to $h(q)$ when t goes to 0.

By applying a suitable affine transformation, we may assume that $q=(0,0)$, the tangent plane T_qM is $z=0$ and the affine normal at q is $(0,1)$. Then, close to q, the surface M is defined by an equation of the form

$$z = \frac{r^2}{2} + O(r^3). \tag{3.2}$$

where $O(r^k)$ may depend on η but satisfies $\lim_{r\to 0}\frac{O(r^k)}{r^{k-\epsilon}}=0$, for any $\epsilon>0$. In this coordinates $h(q)=I$ and $\xi(q)=(0,1)$. Thus we can choose $t=z$ and write $\gamma_q(z) = (\overline{x}(z), z)$.

The following lemma is the main tool for proving theorem 3.1:

Lemma 3.2. *Define*

$$Q_1(z) = \frac{1}{b(z)}\int_{S^{N-1}} r^{N+1}(\eta,z) r_z(\eta,z)\mathbf{M}_N(\eta)d\eta, \tag{3.3}$$

where $b(z)$ denotes the N-volume of the section parallel to the hyperplane $z=0$ at height z. Then

$$Q_1(z) = I + O(z).$$

We now show how theorem 3.1 follows from lemma 3.2. Since $\xi(q)$ is tangent to the centroid line ([8], p.52), we have that $\overline{x}(z) = O(z^2)$. Now from equations (1.2) and (2.3) we conclude that $Q(\gamma_q(z))$ is $O(z^2)$-close to $Q_1(z)$. Hence lemma 3.2 implies that

$$Q(\gamma_q(z)) = I + O(z),$$

thus proving theorem 3.1.

It remains then to prove lemma 3.2.

Proof. Since $\lim_{r\to 0}\frac{r}{\sqrt{2}z^{1/2}}=1$, we can write

$$r(\eta,z)=\sqrt{2}z^{1/2}+O(z^{3/2}). \tag{3.4}$$

Straightforward calculations from (3.4) show that

$$\frac{r^N}{N2^{N/2}}=\frac{1}{N}z^{N/2}+O(z^{N/2+1}).$$

Differentiating $\frac{r^{N+2}}{N+2}$ with respect to z leads to

$$\frac{r^{N+1}r_z}{2^{N/2}}=z^{N/2}+O(z^{N/2+1}).$$

The integral of $\eta_i\eta_j$ over S^{N-1} is equal to $\frac{\lambda}{N}\delta_{ij}$, where $\lambda=\lambda(N)$ is the Lebesgue measure of S^{N-1} and $\delta_{ij}=1$, if $i=j$, and 0, if $i\neq j$. Thus the integral $L(i,j)$ of $r^{N+1}r_z\eta_i\eta_j$ satisfies

$$\frac{L(i,j)}{2^{N/2}}=\frac{\lambda\delta_{ij}}{N}z^{N/2}+O(z^{N/2+1}).$$

Also, calculating $b(z)$ as the integral of r^N/N over S^{N-1} we obtain

$$\frac{b(z)}{2^{N/2}}=\frac{\lambda}{N}z^{N/2}+O(z^{N/2+1}).$$

Thus

$$2^{N/2}b(z)^{-1}=\frac{N}{\lambda}z^{-N/2}+O(z^{-N/2+1}).$$

and so $Q(z)(i,j)=b(z)^{-1}L(i,j)=\delta_{ij}+O(z)$. □

4. Convergence to the shape operator

Along this section, we shall use the notation of [7]: let $f:M\subset\mathbb{R}^N\to\mathbb{R}^{N+1}$ be the inclusion map and denote by ξ its normal vector field pointing to the convex part of M. For $X,Y\in\mathcal{X}(U)$, we write

$$\begin{aligned} D_Xf_*(Y) &= f_*(\nabla_XY)+h(X,Y)\xi\\ D_X\xi &= -f_*(SX), \end{aligned}$$

where ∇ denotes the Blaschke connection, h is the positive definite Blaschke metric and S is the shape operator. Denote by $\nu:M\to\mathbb{R}_{N+1}$ the corresponding co-normal immersion.

Close to the hypersurface M, we write $p=\gamma_q(t)$, $q\in M$, $t\in[0,T)$, where $\gamma_q(t)$ is the centroid of the section through $q+t\xi(q)$ parallel to T_qM. Then p is not necessarily on the normal line $q+t\xi(q)$, but we can write

$$p=q+t\xi(q)+Z, \tag{4.1}$$

for some $Z=Z(q,t)\in T_qM$, with $Z=O(t^2)$ (see [8], p.52). Differentiating (4.1) with respect to t gives

$$\frac{\partial p}{\partial t}=\xi(q)+Z_t, \tag{4.2}$$

for some $Z_t \in T_qM$, with $Z_t = O(t)$. We conclude that

$$v_t(p) = Dv(p) \cdot (\xi(q) + Z_t) = Dv(p) \cdot \xi(q),$$

where for the last equality we have used the orthogonality of $Dv(p)$ and $H(p)$ (see equation (2.6)). We have thus proved the following lemma:

Lemma 4.1. *The derivative of v is given by*

$$Dv(p) = v_t(p)\ \nu(q), \tag{4.3}$$

where $\nu(q)$ is the co-normal vector at $q \in M$ and $v_t(p) = \frac{d}{dt}v(\gamma_q(t))$.

Lemma 4.2. *For any $X \in T_qM$,*

$$\lim_{t\to 0} \frac{1}{v_t} \cdot D^2v(X, \xi) = 0.$$

Proof. Differentiate equation (4.3) with respect to t and use (4.2) to obtain

$$D^2v(\xi(q) + Z_t) = v_{tt}\nu(q).$$

Thus, for any $X \in T_qM$,

$$D^2v(\xi(q) + Z_t, X) = 0.$$

So $D^2v(X, \xi) = -D^2v(X, Z_t)$ and hence

$$\frac{1}{v_t} \cdot D^2v(X, \xi) = Q(\gamma_q(t))(X, Z_t).$$

By corollary 3.1, $Q(\gamma_q(t))$ is converging to h and since $Z_t = O(t)$, we conclude that this last expression converges to 0, thus proving the lemma. □

Theorem 4.3. *The rate of convergence of the bi-linear form $Q(\gamma_q(t))$ to $h(q)$ is $h_S(q)$, i.e.,*

$$\lim_{t\to 0} \frac{Q(\gamma_q(t))(X, Y) - h(q)(X, Y)}{t} = h_S(q)(X, Y).$$

for any $q \in M$, $X, Y \in T_qM$.

Proof. Observe first that if we differentiate (4.1) in the direction $X \in T_qM$, we obtain

$$D_X(p) = (I - tS)X + \nabla_X Z + h(X, Z)\xi(q), \tag{4.4}$$

with $\nabla_X Z = O(t^2)$ and $h(X, Z) = O(t^2)$. Then differentiate equation (4.3) in the direction of $X \in T_xM$ to obtain

$$D^2v(D_X(p)) = v_t\nu_X(q) + X(v_t)\nu(q).$$

Thus, for $Y \in T_qM$,

$$D^2v(D_X(p), Y) = v_t\nu_X(q)(Y) = -v_t h(X, Y)$$

(see [7], p.57, for the last equality). Expanding this equation using (4.4) and dividing by v_t we obtain

$$Q(\gamma_q(t))(I - tSX, Y) - h(X, Y) = -Q(\gamma_q(t))(\nabla_X Z, Y) + h(X, Z)\frac{D^2v(\xi, Y)}{v_t}.$$

Now, from lemma 4.2 and theorem 3.1, we conclude that

$$\lim_{t\to 0}\frac{Q(\gamma_q(t))(X,Y)-h(X,Y)}{t}=h(SX,Y),$$

thus proving the theorem. □

Example. Consider the surface $M\subset\mathbb{R}^3$ described by the equation

$$z=\frac{1}{2}\left(x^2+y^2\right)+\frac{c}{6}\left(x^3-3xy^2\right)+\frac{1}{24}(a_{40}x^4+4a_{31}x^3y+6a_{22}x^2y^2+4a_{13}xy^3+a_{04}y^4).$$

For this surface $\xi(0,0)=(0,0,1)$ and we write

$$z=\frac{r^2}{2}+\frac{r^3}{6}P_3(\theta)+\frac{r^4}{24}P_4(\theta),$$

where $\eta=(\cos(\theta),\sin(\theta),$

$$P_3(\theta)=c\left(\cos^3\theta-3\cos\theta\sin^2\theta\right)=c\cos(3\theta)$$

and

$$P_4(\theta)=a_{40}\cos^4\theta+4a_{31}\cos^3\theta\sin\theta+6a_{22}\cos^2\theta\sin^2\theta+4a_{13}\cos\theta\sin^3\theta+a_{04}\sin^4\theta.$$

It is not difficult to show that, in a neighborhood of $(0,0)$, the inverse function $r=r(z)$ satisfies

$$r(\theta,z)=\sqrt{2}z^{1/2}-\frac{P_3(\theta)}{3}z+\frac{5P_3^2(\theta)-3P_4(\theta)}{18\sqrt{2}}z^{3/2}+O(z^2).$$

From this equation, long but straightforward calculations show that $Q(z)=I+zA+O(z^2)$, where

$$A=\begin{bmatrix}\frac{c^2}{2}-\frac{1}{4}(a_{40}+a_{22}) & -\frac{1}{4}(a_{31}+a_{13})\\ -\frac{1}{4}(a_{31}+a_{13}) & \frac{c^2}{2}-\frac{1}{4}(a_{22}+a_{04})\end{bmatrix}.$$

On the other hand, we can calculate the shape operator of M at the origin following [7], p.47. In this way we verify that $h_S=-A$, in accordance with theorem 4.3.

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